A NOTE ON THE FLUSHING PROCESS IN SALINE SOIL

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In the existing literature on the motion of saline solutions in soil (e.g., [1,2]), it is usually postulated that the velocity of the solution is independent of its concentration. As pointed out in [2], this assumption is valid if the concentration is low. However, in heavily saline soils the concentrations in the flush water may reach values close to the limiting saturation c_* (with common salt, for example, the limiting saturation concentration of a solution at $10°$C is $c_* = 385$ g/l).

The viscosity and density of the solution (mineralized water) increase with an increase in the concentration of the dissolved salts, the viscosity being much more sensitive to changes in mineralization than the density; accordingly, as the concentration varies from zero to the saturation limit, the seepage velocity at a given pressure gradient may fall approximately by a factor of two [3].

Mineralization (g/l):

    0 40-50 60-70 100-110 180-200

Viscosity μ (cP):

    1.0 1.3 1.3-1.4 1.45-1.50 1.8-1.9

Density ρ (g/cm^3):

    1.0 1.03 1.05 1.08 1.15

We assume that the relations between μ and ρ of the solution and c (the concentration) are linear:

$$\mu(c) = \mu_0 + \beta c, \quad \rho(c) = \rho_0 + \alpha c. \tag{1}$$

A satisfactory reflection of the actual $\mu(c)$ and $\rho(c)$ relations can be obtained by means of a suitable choice of coefficients α and β in Eqs. (1).

Considering one-dimensional motion directed vertically downward along the x-axis and neglecting the diffusion effect, we arrive at the equations of the process:

$$v(x, c, t) = -\frac{k}{\mu(c)}\frac{\partial p}{\partial x} + \frac{k\rho(c)g}{\mu(c)}; \tag{2}$$

$$\frac{dv}{dx} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial c}\frac{\partial c}{\partial x} = 0,$$
$$-\frac{\partial(vc)}{\partial x} + \gamma\sigma(c_* - c) = \sigma\frac{\partial c}{\partial t}. \tag{3}$$

Here, $v(x, c, t)$ is the velocity of a solution of concentration $c(x, t)$; k and σ are the permeability and porosity of the soil, respectively (assumed constant); $p(x, t)$ is the pressure in the solution; t is the time; λ is the salt transfer coefficient characterizing the rate at which the salts are dissolved.

Let the pressures at the surface of the soil (x = 0) and at the stationary or moving boundary (x = H or x = h(t) $\in$ [0, H]) be known:

$$p(0, t) = p_0(t), \quad p(h, t) = p_1(t). \tag{4}$$

We write the integral of the first of Eqs. (3) in the form

$$v(x, c, t) = \sigma\frac{dA}{dt}. \tag{5}$$

Here, A(t) is an arbitrary differentiable function subject to determination, with A(0) = 0. Substituting (5) into the second of Eqs. (3), we obtain

$$\frac{\partial c}{\partial t} + \frac{dA}{dt}\frac{dc}{dx} = \gamma(c_* - c). \tag{6}$$

For the case of fresh flush water, we have the initial and boundary conditions:

$$c(0, t), \quad c(x, 0) = f(x). \tag{7}$$

Obviously, in the case of a moving boundary, the initial condition should not be given. The system of characteristics of Eq. (6) is written

thus:

$$\frac{dt}{1} = \frac{dx}{dA/dt} = \frac{dc}{\gamma(c_* - c)}.$$

Integrating this system, with respect to t, from t_0 to t, with respect to x, from x_0 to x, and, with respect to c, from c_0 to c, we find that

$$A(t) - A(t)_0 = x - x_0, \quad t - t_0 = -\frac{1}{\gamma}\ln\frac{c_* - c}{c_* - c_0}. \tag{8}$$

It is easy to see that, in view of conditions (7), the integral surface c = c(x, t) will have a break (or discontinuity, if $f(0) \neq 0$) along the characteristic passing through the point x = t = 0.

Accordingly, setting $x_0 = 0$ and $c_0 = 0$ and eliminating the parammeter t_0, we find

$$t = A^{-1}[A(t) - x] = -\frac{1}{\gamma}\ln\frac{c_* - c}{c_*} \quad (0 \leqslant x \leqslant A(t)).$$

Here, $y = A^{-1}(x)$ is the inverse of the function x = A(y).

Similarly, setting $t_0 = 0$ and $c_0 = f(x_0)$ and eliminating x_0, we obtain

$$t = -\frac{1}{\gamma}\ln\frac{c_* - c}{c_* - f[x - A(t)]} \quad (A(t) < x \leqslant H).$$

Thus, the function c(x, t) can be written in the form

$$c(x, t) = \begin{cases} c_*\{1 - \exp(-\gamma t + \gamma A^{-1})[A(t) - x]\} \\ \quad (0 \leqslant x \leqslant A(t)) \\ c_* - \exp(-\gamma t))[c_* - f(x - A)] \\ \quad (A(t) < x \leqslant H). \end{cases} \tag{9}$$

From (2), we obtain

$$\frac{\partial p}{\partial x} = -\frac{\mu(c)}{k}\sigma\frac{dA}{dt} + g\rho(c).$$

Integrating with respect to x from 0 to H, bearing in mind condition (6), we find that

$$p_1(t) - p_0(t) = -\frac{\sigma}{k}\frac{dA}{dt}\left[\int_0^A \mu(c)\,dx + \int_A^H \mu(c)\,dx\right] + $$
$$+ g\left[\int_0^A \rho(c)\,dx + \int_A^H \rho(c)\,dx\right].$$

Using (1), (2), and (9), we reduce the equation for A(t) to the form

$$\frac{k}{\sigma}\frac{dt}{dA} = \frac{B_1 + \beta(D - E)}{g[B_2 + \alpha(D - E) + p_0 - p_1]}$$

$$B_1 = (\mu_0 + \beta c_*)H - \beta c_* e^{-\gamma t}(H - A),$$

$$B_2 = (\rho_0 + \alpha c_*)H - \alpha c_* e^{-\gamma t}(H - A),$$

$$D = e^{-\gamma t}[F(H - A) - F(0)],$$

$$E = c_* e^{-\gamma t}\int_0^A \exp[\gamma A^{-1}(A - x)]\,dx. \tag{10}$$

Here, $F = \int(x)\,dx$ is a known function. In the case of a moving boundary, A must be substituted for H in Eq. (10). It is easy to see

that Eq. (10) reduces to the following system of first-order differential equations:

$$\frac{k}{\sigma}\frac{dt}{dA} = \frac{B_1 + \beta\,[D - c_*\exp(-\gamma t)\,\Phi(A)]}{g\{B_2 + \alpha\,[D - c_*\exp(-\gamma t)\,\Phi(A)]\} + p_0 - p_1},$$

$$\frac{d\Phi}{dA} = \exp[\gamma t(A)], \quad \Phi(0) = 0, \quad A(0) = 0. \qquad (11)$$

For the initial concentration distribution we assume the linear law

$$f(x) = c(0,0) + \frac{c(H,0) - c(0,0)}{H}\,x.$$

We set $p_1 = p_0$ in system (11) and reduce it to the dimensionless form

$$\frac{d\tau}{dA_1} =$$

$$= \frac{\gamma_1\,(1 + 1/\gamma_1)\exp(\gamma_3\tau) - \Phi(A_1) - (1 - A_1)\,[\delta_1 - \omega(1 - A_1)]}{\gamma_2\,(1 + 1/\gamma_1)\exp(\gamma_3\tau) - \Phi(A_1) - (1 - A_1)\,[\delta_1 - \omega(1 - A_1)]},$$

$$\frac{d\Phi(A_1)}{dA_1} = \exp(\gamma_3\tau), \quad \tau(0) = 0, \quad \Phi(0) = 0.$$

$$A_1 = \frac{A}{H}, \quad \tau = Rt, \quad \gamma_3 = \frac{\gamma}{R}, \quad R = \frac{k}{\sigma}\frac{g\rho_0}{\mu_0 H},$$

$$\gamma_1 = \frac{\beta c_*}{\mu_0}, \quad \gamma_2 = \frac{\alpha c_*}{\rho_0}, \quad \delta_1 = 1 - \frac{c(0,0)}{c_*},$$

$$\omega = \frac{\delta_1 - \delta_2}{2}, \quad \delta_2 = 1 - \frac{c(H,0)}{c_*} \qquad (12)$$

If the salts are only in the liquid phase, then $\gamma_3 = 0$ and the solution of the system will be

$$\Phi(A_1) = A_1$$

$$\tau = \frac{\gamma_1}{\gamma_2}(1 - A^\circ) + \left(\frac{\gamma_1}{\gamma_2} - 1\right)\frac{1}{R} \times$$

$$\times \ln\frac{(2\gamma_2\omega A^\circ + \delta_1^\circ\gamma_2 - R)(2\gamma_2\omega + \delta_1^\circ\gamma_2 + R)}{(2\gamma_2\omega A^\circ + \delta_1^\circ\gamma_2 + R)(2\gamma_2\omega + \delta_1^\circ\gamma_2 - R)}$$

$$(R = \sqrt{(\delta_1\gamma_2)^2 - 4\gamma_2\omega}, \quad A^\circ = 1 - A_1,$$

$$\delta_1^\circ = 1 - \delta_1, \quad \gamma_2\delta_1^2 > 4\omega). \qquad (13)$$

For the case $\omega = 0$, i.e., when the initial concentration distribution is constant with respect to depth, we have

$$\tau = \frac{\gamma_1}{\gamma_2} \times$$

$$\times \left[A_1 + \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)\frac{1}{1 - \delta_1}\ln\left(1 - \frac{A_1}{1 + 1/\gamma_2(1 - \delta_1)}\right)\right]. \qquad (14)$$

Hence, with $A_1 = 1$, we find the total desalinization time.

Curves 1–3 in Fig. 1 illustrate certain numerical and analytical solutions of $\tau(A_1)$ for various parameters: curve 1 corresponds to the case for $\delta_1 = 1$, $\delta_2 = 0$, and $\gamma_3 = 0.5$; curve 2 corresponds to the case for $\gamma_3 = 0$ and $\delta_1 = \delta_2 = 0.5$; curve 3 corresponds to the case for $\delta_1 = 1$,
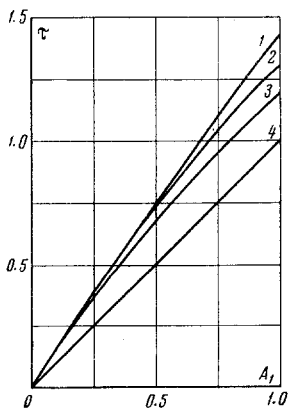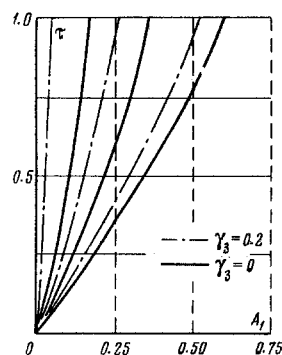

Fig. 1

Fig. 2

$\delta_2 = 0$, and $\gamma_3 = 0$. In these examples $\gamma_1 = 1.6$ and $\gamma_2 = 0.25$. Curve 4 corresponds to the limiting case $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

We now formulate the problem of determining the function $\tau(A_1)$ on the assumption that the saline soil is dry and that the soil air is always located in the closed space between the front and the impervious horizon. We also assume that, in this case, the pressure p_1 of the soil air will increase proportionately as it is compressed by the seeping fluid, i.e., that we can use the equation of state of a perfect gas at constant temperature in the form

$$p_0 H = p_1(H - A) = \text{const}, \quad \text{or} \quad p_0 - p_1 = p_0\frac{A_1}{1 - A_1}. \qquad (15)$$

Since initially the front coincides with the surface of the soil, substituting (13) in (11) and setting H = A and F ≡ 0, we obtain the following system:

$$\frac{d\tau}{dA_1} = \frac{(1 + \gamma_1)\,A_1\exp(\gamma_3\tau) - \gamma_1\Phi(A_1)}{(1 + \gamma_2)\,A_1\exp(\gamma_3\tau) - \gamma_2\Phi(A_1) - h_0 A_0\exp(\gamma_3\tau)},$$

$$\frac{d\Phi}{dA_1} = \exp(\gamma_3\tau), \quad \tau(0) = 0, \quad \Phi(0) = 0,$$

$$A_0 = \frac{A_1}{1 - A_1}, \quad h = \frac{p_0}{\rho_0 g H}. \qquad (16)$$

For physical considerations, it is clear that, given the soil-air trapping condition described, the front will never reach the impervious horizon; its distance from the surface of the soil will tend to the following limit: $\lim A_1(\tau) = \eta$ as $\tau \to \infty$.

Thus, $\lim(dA_1/d\tau) = 0$ as $\tau \to \infty$ and

$$\eta = \frac{1}{1 + h_0/(\gamma_2 + 1)}.$$

In particular, when $\gamma_3 = 0$, the solution of system (14) has the form

$$\tau = \eta_0\left[A_1 - (1 - \eta_0)\ln\left(1 - \frac{A_1}{\eta_0}\right)\right], \quad \eta_0 = \frac{1}{1 + h_0}.$$

Some cases of motion with entrapment are presented in Fig. 2. In this case, the salt concentration in the solution is given by

$$c(x,t) = c_*\{1 - \exp(-\gamma t + \gamma A^{-1})[A(t) - x]\}$$

$$0 \leqslant x \leqslant A(t).$$

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